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EVALUATION OF THE ERROR OF THE METHOD OF SUPERPOSITIONS OF ONE-DIMENSIONAL SOLUTIONS IN NONSTATIONARY HEAT-CONDUCTION PROBLEMS

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Examples of the use of the approximate method in solution of nonstationary heat-conduction problems are given. The error of this method is evaluated by comparison with exact solutions; it turns out to be several orders of magnitude smaller than the error in finite-difference methods. Recommendations on employment of the method are given on the basis of the numerical experiments conducted.

An approximate numerical-analytical method using which eigenfunctions and eigenvalues are found has been proposed in [1]. Having the apparatus of eigenfunctions and eigenvalues, one can obtain solutions for various nonstationary problems that are of importance in heat-conduction theory. We consider the application of this method to determination of eigenfunctions and eigenvalues, which is closely related to solution of the simplest boundary-value problem for a certain curvilinear region Ω with boundary Γ:

$$u_t = a\Delta u\,, \quad u\big|_{\Gamma} = 0\,, \quad u\big|_{t=0} = f(x, y)\,. \tag{1}$$

Using the method of superpositions of one-dimensional solutions for eigenfunctions, we have obtained the integral expression [1]

$$R_i = \int_0^{\pi} [A_i^*(\theta)\cos(\lambda_i \xi(\theta)) + B_i^*(\theta)\sin(\lambda_i \xi(\theta))]\, d\theta\,. \tag{2}$$

If we replace the integral by the integral sum, we will have

$$R_i = \sum_{j=1}^{m} [A_{ij}\cos\lambda_i\xi_j + B_{ij}\sin\lambda_i\xi_j]\,. \tag{3}$$

The eigenfunctions and eigenvalues R_i and λ_i are found from the homogeneous integral equation

$$\int_0^{\pi} [A_i^*(\theta)\cos(\lambda_i\xi_{\Gamma}) + B_i^*(\theta)\sin(\lambda_i\xi_{\Gamma})]\, dt = 0\,, \quad \xi_{\Gamma} = (\mathbf{r}_{\Gamma} - \mathbf{r}_0)\,\mathbf{n}\,, \tag{4}$$

where $\mathbf{r}_{\Gamma}$ is the radius vector of the points of the boundary Γ, or from the analog of Eq. (4) represented by the integral sum and having the form of a homogeneous system of linear algebraic equations for A_{ij} and B_{ij}:

$$\sum_{j=1}^{m} [A_{ij}\cos\lambda_i\xi_{jk} + B_{ij}\sin\lambda_i\xi_{jk}] = 0\,, \quad k = 1, V, ..., 2m\,, \quad i = 1, V, ..., \infty\,, \tag{5}$$

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here $\xi_{ik} = (\mathbf{r}_k - \mathbf{r}_0)\mathbf{n}_j$, $\mathbf{r}_k$ being the radius vectors of the points of division of the boundary Γ into small parts. By solution of system (5), we find λ_i, A_{ij}, and B_{ij}. Substituting them into (3), we obtain R_i; then the solution of problem (1) can be represented by the expression

$$u = \sum_{i=1}^{\infty} C_i R_i \exp(-a\lambda_i^2 t), \quad C_i = \left[\iint_{\Omega} f(x, y)\, R_i ds\right] \Big/ \left[\iint_{\Omega} R_i^2\, ds\right]. \tag{6}$$

Thus, for construction of the approximate solution (6) we must precompute the spectra $\{\lambda_i\}$ and $\{R_i\}$ using (3) and (5). It is seen from formula (6) that the accuracy of computation of u is determined by the error for λ_i and R_i, which rapidly decreases with increase in t. Therefore, in what follows we will evaluate the error only for λ_i and R_i and partial derivatives of R_i of different order with respect to the coordinate x.

If we consider problem (1) for a plane wall ($0 \le x \le h$), the solution found by this method will exactly coincide with the solution obtained by the Fourier method. We investigate (1) for the region of a regular triangle with height h when $f(x, y) = 1$. For this case we know the exact solution [2]:

$$u = \sum_{n=1}^{\infty} \frac{2}{\pi n}\left[\sin 2\pi n \frac{\xi_1^*}{h} + \sin 2\pi n \frac{\xi_2^*}{h} + \sin 2\pi n \frac{\xi_3^*}{h}\right] \exp\left[-\left(\frac{2n\pi}{h}\right)^2 at\right],$$

$$\xi_i^* = (\mathbf{r} - \mathbf{r}_i^*)\,\mathbf{n}_i^*, \quad i = 1, 2, 3. \tag{7}$$

According to the method, the number of division points at the boundary of the triangle Γ must be even; furthermore, it is divisible by three because of the triple symmetry. Therefore, the taken number of division points should be divisible by six and that of the rays E should be divisible by three. Numerical experiments have shown that if, among the set of rays E, there are three rays directed perpendicularly to the sides of the triangle, in this case the approximate solution will coincide with the exact solution from (7).

From the above two cases it follows that sometimes this approximate method allows exact solutions, whereas all the existing finite-difference methods lead to an approximate solution with the possibility of computing only low-order partial derivatives of the functions sought [3].

We can make the following statement: if we are able to solve the homogeneous integral equation (4), the superposition method yields the exact solution (6). The exactness is determined by the error in replacement of integrals (2) and (4) by the integral sums (3) and (5). Let us demonstrate this, using as an example problem (1) on cooling of a solid cylinder of radius $r_0 = 1$ with an initial temperature of $f(x, y) = 1$. The problem has the following exact solution [4]:

$$u = 2 \sum_{j=1}^{\infty} \frac{J_0(\lambda_j r / r_0)}{\lambda_j J_1(\lambda_j)} \exp\left(-\frac{a\lambda_j^2}{r_0^2} t\right). \tag{8}$$

In constructing the approximate solution of (3) for the eigenfunctions R_i, we must select a computational scheme consisting of the pole $\mathbf{r}_0$, the system of rays E drawn through the pole at angles θ_i to the x axis, and the system of points $\mathbf{r}_k$ by which the boundary Γ is divided into small parts.

The origin of coordinates (x, y) will be located at the center of a circle where the pole $\mathbf{r}_0 = 0$ will find itself. To simplify the computations we divide the circle into p equal sectors by the rays E. In this problem, for equal distance from the origin of coordinates r on each ray the eigenfunctions R_i must take equal values, namely:

$$A_i^*(\theta) = A_i^* = \text{const}, \quad B_i^*(\theta) = B_i^* = \text{const}, \quad A_{1i} = A_{2i} = \ldots = A_{mi} = A_i, \quad B_{1i} = B_{2i} = \ldots = B_{mi} = B_i. \tag{9}$$

Furthermore, the diametrically opposite points M^+ and M^- of the circle region correspond to the values of ξ with opposite signs, i.e., $\xi(M^+) = -\xi(M^-)$. In this case we have $R_i(M^+) = R_i(M^-)$; therefore, the coefficients before $\sin(\lambda_i \xi(\theta))$ and $\sin(\lambda_i \xi_j)$ in expressions (2) and (3) must be equal to zero:

TABLE 1. Error for the Eigenvalues as a Function of the Number of Computational Points

m	$\delta\lambda_1$	$\delta\lambda_2$	$\delta\lambda_i$
20	10^{-45}	10^{-30}	$\max i = 8,\ \delta\lambda_8 = 10^{-6}$
30	10^{-77}	10^{-55}	$\max i = 13,\ \delta\lambda_{13} = 10^{-6}$
50	10^{-150}	10^{-114}	$\max i = 24,\ \delta\lambda_{24} = 10^{-6}$

TABLE 2. Error for the Eigenfunctions and Partial Derivatives as a Function of the Number of Computational Points

m	δR_1	δR_2	δR_{1x}	δR_{2x}	δR_{1xx}	δR_{2xx}	δR_{1xxxx}	δR_{2xxxx}
20	10^{-45}	10^{-30}	10^{-43}	10^{-29}	10^{-42}	10^{-27}	10^{-39}	10^{-24}
50	10^{-150}	10^{-114}	10^{-148}	10^{-112}	10^{-146}	10^{-110}	10^{-142}	10^{-106}

$$B_i^*(\theta) = B_{1i} = B_{2i} = \ldots = B_{mi} = B_i = 0 . \tag{10}$$

The results of relations (9) and (10) can also be obtained numerically by solution of system (5) without any preliminary assumptions. According to the formulas $x = r\cos\varphi$ and $y = r\sin\varphi$, we pass to a polar coordinate system

$$\xi = r\cos(\theta - \varphi) , \quad \xi_\Gamma = r_0\cos(\theta - \varphi) , \quad \xi_j = r\cos(\theta_j - \varphi) , \quad \xi_{jk} = r_0\cos(\theta_j - \varphi_k) . \tag{11}$$

Using (11) we transform expressions (2) and (3):

$$R_i = A_i\int_0^\pi \cos[\lambda_i r\cos(\theta - \varphi)]\,d\theta , \quad R_i = A_i\sum_{j=1}^{m}\cos[\lambda_i r\cos(\theta_j - \varphi)] . \tag{12}$$

By replacement of the variable $\theta - \varphi = \tau$, the expressions for R_i can be transformed to the form

$$R_i = A_i\int_0^\pi \cos(\lambda_i r\cos\theta)\,d\theta , \quad R_i = A_i\sum_{j=1}^{m}\cos(\lambda_i r\cos\theta_j) . \tag{13}$$

Let us use one integral form of representation of the Bessel function [5]:

$$J_0(z) = \frac{1}{\sqrt{\pi}\,\Gamma(1/2)}\int_0^\pi \cos(z\cos t)\,dt . \tag{14}$$

After the comparison of (12) and (14), we obtain that the solution in the form (2) exactly coincides with (8) and the equation $J_0(\lambda r) = 0$ for finding the eigenvalues λ_i coincides with (4) when (9) and (10) hold. Therefore, expressions (3) and (5) are their approximate analogs.

The results of numerical solution of system (4) with an accuracy of calculation of 200 decimal places after the point that have been obtained without allowance for any simplifying assumptions on the properties of solution for a circle are given in Tables 1 and 2. The errors of the eigenfunctions δR_i and their partial derivatives have been determined at Γ at the centers between division points. It is approximately at such points that the errors δR_i take the highest value. At the internal points of Ω, the errors decrease and the accuracy is further improved.

Table 1 gives the errors for eigenvalues $\delta\lambda_i$ as a function of m. It follows from the table that when 50 rays and accordingly 100 points of division of Γ are employed, we can compute 24 eigenfunctions λ_i and R_i with a high degree of accuracy. Whereas the errors are very small for λ_1 and R_1, the errors $\delta\lambda_i$ and δR_i monotonically increase

with increase in the number i. However, this drawback is partially suppressed, since the influence of the eigenfunction R_i decreases by virtue of the convergence of spectral decompositions as i increases.

We can infer that when $2m$ points of division of Γ, where $m > 30$, are employed, this method allows computations of approximately $m/2$ eigenvalues and eigenfunctions with an accuracy no lower than 10^{-6}. As m increases, all the errors rapidly decrease; thus, we have $\approx 10^{-3m}$ for $\delta\lambda_1$. Not only does selection of m depend on the error δR_i but it also depends on the number of terms in the spectral decomposition that are required for maintaining the prescribed accuracy.

Table 2 gives the errors for partial derivatives at the boundary points on the x axis. It is clear from the table that each two partial derivatives increase the error by approximately three orders of magnitude. Nonetheless, the errors are very small even for the fourth partial derivative. The influence of the location of the pole $\mathbf{r}_0$ on the computational error is slight.

The numerical experiments have shown that if no assumptions on the symmetry of solution for the circle region are made, i.e., if the general case is considered, the eigenfunctions and eigenvalues corresponding to all orders of the Bessel functions of the first kind are automatically determined from system (5). Spectral decomposition of the initial condition of problem (1) itself will "select" necessary eigenfunctions orthogonal in Ω, which has also been checked.

It is clear from the given tables that the accuracy of the method is high and we can compute partial derivatives of high order with it. Furthermore, the computational process in this method is much less labor-consuming than that in finite-difference schemes. Also, it is noteworthy that the approximate solution here has an analytical form and exactly satisfies the initial differential equation, the initial condition, and the boundary conditions at the points of division of the boundary Γ into small parts. The boundary conditions hold approximately just at the points of the boundary between the division points.

Based on the numerical experiments, we have obtained the following recommendations on application of this method:

1. To implement the method one must first select a computational scheme that consists of the pole $\mathbf{r}_0$, the rays E, and the points $\mathbf{r}_k$ of division of the boundary Γ of a given region Ω into small parts.

2. The location of the pole $\mathbf{r}_0$ relative to the region Ω for which the solution is sought has no influence, in practice, on the value of the error.

3. To compute the eigenvalues λ_i one investigates the function $y = \Delta_{2m}(\lambda)$ representing the determinant of system (5). The points of intersection of the λ axis are the roots of the equation $\Delta_{2m}(\lambda) = 0$. For a small number $2m$ of points of division of Γ, when information on the shape of the region Ω is insufficient, Eq. (14) will have false roots whose characteristic feature is their mobility, i.e., they significantly shift along the λ axis with increase in $2m$. For each region Ω we have a certain minimum number $2m_1$ of points of division of Γ, when the first reliable eigenvalue λ_1 appears (with a certain error), which depends on the shape and dimensions of Ω. A characteristic property of this root of λ_1 is its relative stability, i.e., several figures of this root remain constant with increase in m, whereas in the interval $(0, \lambda_1)$, there are no false moving roots. For any m we have a certain finite number N of eigenvalues such that $\lambda_1, V, ..., \lambda_n$ are reliable. In addition to the stability of the values, similar eigenvalues have one more characteristic property: there are no false moving roots between any two eigenvalues λ_{i-1} and λ_i $(1 \le i \le N)$.

4. If the curvilinear boundary Γ has a rectilinear portion, the accuracy of the method is improved when one ray E is directed along the normal to the rectilinear portion.

5. If the boundary Γ has angular points, they must be included in the system of division points.

NOTATION

a, thermal diffusivity, m^2/sec; A_i^* and B_i^*, unknown functions; A_{ij} and B_{ij}, unknown coefficients; C_i, coefficients of special decomposition; $f(x, y)$, initial temperature, K; h, height of the triangle, m; J_0 and J_1, Bessel functions of the first kind; M^+ and M^-, diametrically opposite points of the circle; m, number of rays E; n, natural number; $\mathbf{n}$ and $\mathbf{n}_j$, unit vectors; $\mathbf{n}_i^*$, internal unit vectors to the sides of the triangle; $\mathbf{r}$, $\mathbf{r}_0$, $\mathbf{r}_\Gamma$, and $\mathbf{r}_k$, radius vector of an arbitrary point from Ω, of the pole, of arbitrary points at Γ, and of the points of division of Γ into small parts, m; $\mathbf{r}_i^*$, radius vectors of the vertices of the triangle, m; r, polar radius, m; r_0, radius of the cylinder, m; t, time, sec; u, temperature,

K; x and y, Cartesian coordinates; z, auxiliary variable; Ω and Γ, region and its boundary; Δ_{2m}, determinant of the system of $2m$ linear equations; $\Gamma(1/2)$, gamma function of 1/2; $\delta\lambda_i$ and δR_i, errors of computations of the eigenvalues and eigenfunctions; θ, slope of the rays E to the x axis; λ_i and R_i, eigenvalues and eigenfunctions; ξ, ξ_j, and ξ_{jk}, special variables, m; ξ_1^*, ξ_2^*, and ξ_3^*, special variables for the regular triangle, m; φ, polar angle. Subscripts: i, Nos. of eigenvalues and eigenfunctions; j, Nos. of terms in the finite sum; k, Nos. of points of division of the boundary Γ.

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